Lattice Formation in a WAdL (**W**ireless **Ad**-hoc **L**attice Computer)

Vishakha Gupta, Gaurav Mathur
Birla Institute of Technology & Science
Pilani, India
email: vishakhasgupta@hotmail.com, gauravjsr@hotmail.com

Anil M. Shende
Roanoke College
Salem, VA, USA
email: shende@roanoke.edu

Abstract

We propose an architecture to harness the comparatively low computational power of geographically concentrated mobile devices (such as in a wireless ad hoc network, especially a sensor network) to build a wireless ad hoc lattice computer (WAdL). WAdL is a cellular automaton-like architecture designed to analogically simulate the unfolding of a physical phenomenon (e.g., fluid flow, system of moving, interacting objects, etc.) in the bounded region of euclidean space represented by the underlying virtual lattice of WAdL.

In this paper we present a constant time algorithm for the mobile devices in a given geographic area to form a wireless ad-hoc (2-dimensional) lattice computer.

1 Introduction

Scientific computing largely deals with the prediction of attribute values of objects participating in physical phenomena. Usually the unfolding of these phenomena involves motion of the participating objects. There are no known analytical solutions for some physical phenomena, and the only apparent method of prediction is the *analogical* simulation of the unfolding of the phenomenon in a cellular automaton-like architecture — a *lattice computer* — representing the region of euclidean space in which the phenomenon unfolds [7]. Lattice computers are massively parallel machines where the processing elements are arranged in the form of a regular grid, and where the computational demand on each individual processing element is quite low [8, 3, 9]. Each processing element represents a point/region of euclidean space. In analogical simulations on a lattice computer, the motion of an object across euclidean space is carried out as a sequence of steps, uniform in time, where in each step the representation of the object may move from one processing element to a *neighbour*, as defined by the underlying grid of the lattice computer [11].

The proliferation of portable, wireless computing devices (e.g., cell phones, PDAs) promises the availability of a large number of computing devices in a relatively small geographic region. Much research has been focused on creating ad-hoc networks using such mobile devices, and the applications for such networks are primarily for individual gain by end users (e.g., e-commerce for the owner of a device in the network). When such devices are equipped with sensors, the resulting wireless sensor networks are typically used for data acquisition; each sensor collects data from its surroundings and conveys that information to a central processing unit for analysis and for initiating some action.

Such an ensemble of wireless devices (computing devices and/or sensors), despite their limited computational capacity, and limited range of communication, provides a rich infrastructure for creating a wireless ad-hoc lattice computer (WAdL). We propose the WAdL architecture as a wireless ad-hoc distributed computing environment for harnessing the collective computing capabilities of the devices for the common cause of scientific computing.

In this paper we first give a brief overview of our proposed architecture for a WAdL[1] and then focus on the formation of a lattice from a collection of mobile devices in a given geographic region. In particular, we show that such a lattice can be formed in constant time.

The rest of this paper is organised as follows: We discuss analogical simulations in more detail in Section 2. Section 3 describes in brief the architectural frame-work of WAdL. In Section 4 we develop the mathematical basis for lattice formation, and thus show that the underlying lattice for a WAdL can be formed in constant time.

2 Analogical Simulations

A physical phenomenon is a development in a region of euclidean space over a period of time. At each instant in time (in a given time period), the set of objects participating in the phenomenon, together with their attribute values (such as speed, spin, etc.) at that time, completely describes a snapshot in the unfolding of the phenomenon. Most problems in scientific computing are about phenomena whose unfolding involves the motion of participating objects in euclidean space. Solutions to these phenomena usually involve determining (predicting) the attribute values of objects over time. Some phenomena can be solved analytically using closed form functions of time. On the other hand, there are phenomena where the only apparent method for predicting the attribute values of participating objects at any instant in time, is to *simulate* the unfolding of the phenomenon up through that instant of time [5, 7].

When carried out on a digital computer such simulations, *necessarily*, develop in a discretized representation of a region of euclidean space, and over discrete time units. Moreover, such simulations must use, at any given instant of simulation time, only information available *locally*, at a discrete point in the represented euclidean space, to compute the attribute values of participating objects at the next instant of simulation time. Cellular automaton machines [9] and lattice computers [3] provide the necessary framework for a discretized representation of euclidean space in which to carry out such simulations. Several physical phenomena, including spherical wavefront propagation [2] and fluid flow [6, 12], have been successfully simulated on such a framework where the simulation algorithms do *not* use the traditional analytical models for the phenomena.

3 The WAdL Architecture

The goal of the WAdL architecture is to use the participating mobile devices (henceforth referred to as *nodes*) in a geographic area to discretize a bounded region of euclidean space, B, and then carry out analogical simulations of physical phenomena in this discretized representation of B using the methodology proposed in [3]. The model described in [3] uses a *root* lattice as the discretized representation of a bounded region of euclidean space. There is a processor at each lattice point, and this processor, say at lattice point p, can communicate with only its neighbours in the lattice, i.e., those processors at lattice points q such that $(q - p)$ is a minimal length vector in the lattice. Each of these processors is required to have minimal computational ability.

Hence, in a WAdL, the participating nodes (mobile devices in the given geographical region) are logically organised so that each node is responsible for a point of a bounded piece of a root lattice L; This piece of L represents B.[2] The logical organisation of the nodes is such that each node, responsible for a lattice point p, is within communication range of all nodes responsible for any lattice point q where $(q - p)$ is a minimal length vector in the lattice.

We assume that each participating node has some, minimal computational and storage facilities, is equipped with some form of location ser-

[1] A paper describing in detail our proposed WAdL architecture, including mechanisms for handling fault tolerance, and presenting a simulation of the proposed architecture for an application is currently under review for publication.

[2] A lattice, by definition, is an infinite object. Nonetheless, for expositional convenience, henceforth in this paper, by lattice we will mean a finite piece. Thus, it is reasonable for a lattice to represent a bounded region of euclidean space, to count the number of points in a lattice, etc.

vice, e.g., GPS, and has some communication capabilities, e.g., Bluetooth.

A WAdL consists of a single immobile node or a base-station (denoted by I) designated as the manager, and a collection of mobile devices as the nodes in the lattice computer. I fixes a lattice, L, with a fixed origin, in its region of influence, and then, each lattice point, p, is assigned all mobile devices within the *Voronoi cell*[3] around p. Note that several devices may be assigned to the same lattice point. Section 4 provides details on this assignment of devices to lattice points.

Once a WAdL is formed, I initiates the simulation of a physical phenomenon in the underlying lattice computer formed from the mobile devices in B. Since several devices may be assigned to one lattice point, p, they elect a leader primarily responsible for p; all the other devices assigned to p also carry out the computations performed by the leader, so that if and when the leader moves out of the Voronoi cell of p, another device assumes leadership and the overall simulation can continue seamlessly.

Depending on the application, it may be necessary to perform the simulation in a virtual root lattice V whose dimension is higher than that of the lattice L formed from the devices in the region. In this case, the application provides the mechanism for establishing the correspondence between L and V, and hence, in this paper we focus only on the formation of a 2-dimensional lattice L.

4 Lattice Formation

As noted in the previous section we will focus on 2-dimensional root lattices, and hence, we will restrict our attention to the square lattice ($\mathbf{Z}^2$), and the cellular lattice ($\mathbf{A}_2$), the only root lattices in 2-dimensional euclidean space. (Technically, $\mathbf{Z}^2$ is the square lattice with minimal distance 1, and $\mathbf{A}_2$ is the cellular lattice with minimal distance $\sqrt{2}$; for expositional convenience, by $\mathbf{Z}^2$ and $\mathbf{A}_2$ we simply mean lattices that are generated by equal, orthogonal basis vectors, and equal basis vectors at an angle of $\pi/3$ to each other, respectively; we will denote the minimal distance in the particular lattice we use by $\mu(\mathbf{Z}^2)$ and $\mu(\mathbf{A}_2)$, respectively.)

[3]The *Voronoi cell around a lattice point* p, by definition, is the set of points t in euclidean space such that t is closer to p than to any other lattice point.

4.1 Lattice Resolution

In order to map each mobile device to L, we need to determine the resolution of L, i.e., the minimal distance between points in L, denoted by $\mu(L)$. Each mobile device in the region will be assigned to a point, p, of L, such that the mobile device is in the Voronoi cell around p in L. Clearly, then, $\mu(L)$ must be chosen such that any two mobile devices that are assigned to adjacent points in L can communicate with each other. The following lemma establishes a reasonable upper bound on $\mu(L)$.

Lemma 1. *Suppose the underlying lattice for a WAdL is L. Suppose the range of the transmitter/receiver of all devices in the region is at least ρ. Then, devices that are assigned to adjacent points in L can communicate with each other if*

$$\mu(L) \leq \begin{cases} \frac{\sqrt{2}\rho}{2+\sqrt{2}}, & \text{if } L = \mathbf{Z}^2 \\ \frac{\sqrt{3}\rho}{2+\sqrt{3}}, & \text{if } L = \mathbf{A}_2. \end{cases}$$

Proof. We will first establish an elementary fact regarding distances between points in two spheres.

Claim 1. *Suppose C_1 and C_2 are two closed circles, centered at c_1 and c_2, and with radii r_1 and r_2, respectively. Then, for any two points p in C_1 and q in C_2, $d(p,q) \leq d(c_1,c_2) + r_1 + r_2$.*

Proof. [Claim 1] Using the traingle inequality, it follows that

$$\begin{aligned} d(p,q) &\leq d(p,c_2) + d(c_2,q) \\ &\leq d(c_1,c_2) + d(c_1,p) + d(c_2,q) \\ &\leq d(c_1,c_2) + r_1 + r_2. \end{aligned}$$

□ Claim 1

Voronoi cells around any lattice point are simply translates of the Voronoi cell around the origin of the lattice. Each face of the Voronoi cell around the origin perpendicularly bisects some vector in the lattice. These vectors are called *relevant vectors* in the lattice. (See [4] for more on Voronoi cells in lattices.) Both $\mathbf{Z}^2$ and $\mathbf{A}_2$, being root lattices, have the property that the set of minimal length vectors is the same as the set of relevant vectors, and hence, the Voronoi cells in these lattices are circumscribed by spheres [10]. It can be easily verified that in the case of $\mathbf{Z}^2$, the radius of this circumscribing sphere is $\frac{\mu(\mathbf{Z}^2)}{\sqrt{2}}$, and in the case of $\mathbf{A}_2$, it is $\frac{\mu(\mathbf{A}_2)}{\sqrt{3}}$.

Clearly, every point in the Voronoi cell around a lattice point is in the sphere circumscribing the Voronoi cell. Let t be the distance between two devices assigned to adjacent lattice points. Then, using Claim 1, we have an upper bound on t:

$$t \leq \begin{cases} \mu(L) + 2 \cdot \frac{\mu(L)}{\sqrt{2}}, & \text{if } L = \mathbf{Z}^2 \\ \mu(L) + 2 \cdot \frac{\mu(L)}{\sqrt{3}}, & \text{if } L = \mathbf{A}_2. \end{cases}$$

The two devices will be able to communicate if the transmission range, ρ, of the weakest device is more than this upper bound, i.e.,

$$\rho \geq \begin{cases} \mu(L) + 2 \cdot \frac{\mu(L)}{\sqrt{2}}, & \text{if } L = \mathbf{Z}^2 \\ \mu(L) + 2 \cdot \frac{\mu(L)}{\sqrt{3}}, & \text{if } L = \mathbf{A}_2. \end{cases}$$

Thus, solving for $\mu(L)$, we have the lemma. □

4.2 Forming the Lattice

As noted in Section 3, we assume that all the devices participating in the WAdL are equipped with GPS. We will assume that the fixed device, the base-station for the WAdL, and the mobile devices use the Earth-Centered, Earth-Fixed (ECEF) coordinate system (euclidean coordinate system, where the the center of the earth is the origin, the X axis is defined by the intersection of the equator and the 0 longitude, and the Y axis is defined by the intersection of the equator and the 90E longitude. [1]).

In a WAdL, the coordinates, I, of the base-station are the origin of the underlying lattice, L. The lattice L is completely specified by the origin and a *basis* for the lattice. Since we are concerned only with 2-dimensional lattices, the basis for our lattice will have a size of two.

Definition 1. *A basis $\{b_1, b_2\}$ for a 2-dimensional lattice L is a* regular *basis iff*

1. $||b_1|| = ||b_2||$, *and*

2. $$(b_1, b_2) = \begin{cases} 0, & \textit{if } L = \mathbf{Z}^2 \\ \frac{||b_1||^2}{2}, & \textit{if } L = \mathbf{A}_2. \end{cases}$$

In other words, if L is the square lattice, the basis vectors in a regular basis for L are equal in length and at an angle of $\pi/2$ to each other; if L is the cellular lattice then the basis vectors in a regular lattice for L are equal in length and at an angle of $\pi/3$ to each other. Note that, the length of each of the basis vectors, in these cases, is also the length of a minimal length vector, $\mu(L)$, in the lattice.

Based on the weakest transmitting device at the time of first forming the WAdL, the base-station computes the resolution of L, and hence, the vectors, b_1 and b_2 such that $\{b_1, b_2\}$ is a regular basis for L. Each mobile device is sent I, these basis vectors, and the type of lattice ($\mathbf{Z}^2$ or $\mathbf{A}_2$) to be formed. We now discuss how each mobile device can then compute the lattice point in L that it is assigned to.

In this paper, we assume that all the mobile devices are, indeed, in the plane of the lattice defined by b_1 and b_2. This is a reasonable assumption to make in some scenarios, e.g., where the participating mobile devices are sensors placed in an open field. Dealing with the case where some of the mobile devices may not be in the same plane as the underlying lattice is an intriguing problem for further research.

For the lattice L, let $M(L) = \{m_1, m_2, ..., m_k\}$ denote the set of minimal length vectors. The set of minimal length vectors for our two lattices are then given by

$$\begin{aligned} M(\mathbf{Z}^2) &= \{b_1, b_2, -b_1, -b_2\}. \\ M(\mathbf{A}_2) &= \{b_1, b_2, -b_1, -b_2, b_1 - b_2, b_2 - b_1\}. \end{aligned}$$

Definition 2. *A* positive representation *for a point p in the lattice L is a four-tuple $\langle \alpha, \beta, m_i, m_j \rangle$ such that:*

1. $\alpha, \beta \geq 0$,

2. $\{m_i, m_j\} \subseteq M(L)$, *and*

3. $p = \alpha m_i + \beta m_j$.

We will sometimes write the positive representation for a point p in L as $\alpha m_i + \beta m_j$.

Every point in the plane of the underlying lattice of a WAdL is either along the line defined by a minimal length vector, or is in the open cone bounded by the lines defined by two minimal length vectors that are orthogonal in the case of $\mathbf{Z}^2$, or at an angle of $\pi/3$ to each other in the case of $\mathbf{A}_2$. Thus it is easy to verify that

Lemma 2. *Suppose $L \in \{\mathbf{Z}^2, \mathbf{A}_2\}$. Then, every point, p, in the plane of L has a positive representation $\langle \alpha, \beta, m_i, m_j \rangle$ such that $\{m_i, m_j\}$ is a regular basis for L.*

Now, Lemma 3, below, follows from the fact that adjacent minimal length vectors are orthogonal in $\mathbf{Z}^2$ and are at an angle of $\pi/3$ to each other in $\mathbf{A}_2$. Since the number of minimal length vectors in each case is finite and computable, this lemma effectively provides an algorithm for computing a positive representation for any point.

Lemma 3. *Suppose D is a point in the plane of L and $M(L) = \{m_1, \ldots, m_k\}$. Then,*

1. $D = \alpha m_i, \alpha > 0$ *iff* $\frac{(D,m_i)}{(\|D\|\mu(L))} = 1$, *and*
2. $D = \alpha m_i + \beta m_j, \alpha, \beta > 0$, *and* $\{m_i, m_j\}$ *is a regular basis for L if*
   (a) $\frac{(D,m_i)}{(\|D\|\mu(L))} > 0$ *and* $\frac{(D,m_j)}{(\|D\|\mu(L))} > 0$, *if* $L = \mathbf{Z}^2$, *and*
   (b) $\frac{(D,m_i)}{(\|D\|\mu(L))} > \frac{1}{2}$ *and* $\frac{(D,m_j)}{(\|D\|\mu(L))} > \frac{1}{2}$, *if* $L = \mathbf{A}_2$.

□

Lemmas 4, 5 and 6 provide algorithms for determining the lattice point p, such that a given point D in the plane of the lattice in in the Voronoi cell around p. Lemmas 4 and 5 are fairly straightforward; we provide a sketch of a proof for Lemma 6

Lemma 4. *Suppose the positive representation for a point D is αm_i. Then, D is in the Voronoi cell around the lattice point $rnd(\alpha)m_i$.* □

Lemma 5. *Suppose $L = \mathbf{Z}^2$, and suppose the positive representation for a point D is $\alpha m_i + \beta m_j$. Then, D is in the Voronoi cell around the lattice point $rnd(\alpha)m_i + rnd(\beta)m_j$.* □

Lemma 6. *Suppose $L = \mathbf{A}_2$, and suppose the positive representation for a point D is $\alpha m_i + \beta m_j$. Let*

$$\begin{aligned} P &= \lfloor\alpha\rfloor m_i + \lfloor\beta\rfloor m_j \\ Q &= \lceil\alpha\rceil m_i + \lfloor\beta\rfloor m_j \\ R &= \lfloor\alpha\rfloor m_i + \lceil\beta\rceil m_j \\ D_1 &= D - P, \text{ and} \\ D_2 &= D - Q. \end{aligned}$$

(See Figure 1.) Then, D is in the Voronoi cell around the lattice point

1. *P, if* $\frac{(D_1,m_i)}{\mu^2} \leq \frac{1}{2}$ *and* $\frac{(D_1,m_j)}{\mu^2} \leq \frac{1}{2}$,
2. *Q, if* $\frac{(D_1,m_i)}{\mu^2} > \frac{1}{2}$ *and* $\frac{(D_1,m_j)}{\mu^2} \leq \frac{1}{2}$,
3. *R, if* $\frac{(D_1,m_i)}{\mu^2} \leq \frac{1}{2}$ *and* $\frac{(D_1,m_j)}{\mu^2} > \frac{1}{2}$,
4. *Q, if* $\frac{(D_1,m_i)}{\mu^2} > \frac{1}{2}$ *and* $\frac{(D_1,m_j)}{\mu^2} > \frac{1}{2}$, *and* $\frac{(D_2,m_j-m_i)}{\mu^2} \leq \frac{1}{2}$.
5. *R, if* $\frac{(D_1,m_i)}{\mu^2} > \frac{1}{2}$ *and* $\frac{(D_1,m_j)}{\mu^2} > \frac{1}{2}$, *and* $\frac{(D_2,m_j-m_i)}{\mu^2} > \frac{1}{2}$,

Proof. The expressions above compute the projection of the point $D - P$ on m_i and m_j. For example, Item (1) above computes the lengths of the projections on both m_i and m_j, and if the lengths of both the projections are no more than half the minimal length, then D is in the Voronoi cell around P. Similarly Items (3) and (2). Items (4) and (5) deal with the case where the projections of $D - P$ on both m_i and m_j are more than half the minimal length. In that case, D is in the Voronoi cell around Q or R, and this decision is based on the projection of $D - Q$ on $m_j - m_i$. (For example, the point D shown in Figure 1 will satisfy the conditions in Item (5), and thus is in the Voronoi cell around R.) □

Now, suppose a participating mobile device is at coordinates C. It first computes it's coordinates $D = C - I$, relative to the origin of the lattice. Then, using the algorithm provided by Lemma 3, it computes a positive representation for D. Finally, using one of the Lemmas 4, 5 and 6, as appropriate, it computes the lattice point it is assigned to.

Thus, based on the computations involved in the above lemmas, we have the following theorem:

Theorem 1. *Each participating mobile device in a WAdL can compute the lattice point it is responsible for in* constant *time.* □

5 Conclusion

We propose an architecture to utilize the geographical concentration, within a bounded region B, of mobile devices in a wireless ad hoc network to construct a wireless ad-hoc lattice computer (WAdL), that could be used to perform scientific

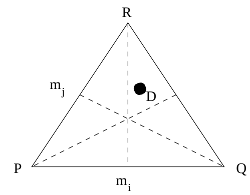

Figure 1: Computing closest lattice point in $\mathbf{A}_2$

analogical simulations. In the WAdL, a base station manages the formation of the wireless ad-hoc lattice computer from the participating devices. In this paper we have presented an algorithm to have each device, entering the area of influence of the base station, compute the lattice point, in the underlying 2-dimensional lattice, it is responsible for. We have also showed that this computation can be performed in constant time regardless of the size of B. As mentioned in Section 4.2, extending our results to form a 3-dimensional root lattice is an intriguing open problem.

We note here that the formation of a virtual lattice out of a collection of mobile devices can also be used effectively for routing purposes in the resulting wireless ad-hoc network. The rich structure of the underlying lattice will also readily provide several paths between pairs of devices, and thus could be used to address congestion issues.

References

[1] Glossary of GPS terms (http://tsc.wes.army.mil/gps/glossary.htm).

[2] John Case, Dayanand Rajan, and Anil M. Shende. Spherical wavefront generation in lattice computers. In *Proceedings of the 6th International Conference on Computing and Information*, May 1994.

[3] John Case, Dayanand S. Rajan, and Anil M. Shende. Lattice computers for approximating euclidean space. *Journal of the ACM*, 48(1):110–144, 2001.

[4] J. Conway and N. Sloane. *Sphere Packings, Lattices and Groups*. Springer Verlag, second edition, 1993.

[5] Richard P. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21(6/7), 1982.

[6] U. Frisch, B. Hasslacher, and Y. Pomeau. Lattice-gas automata for the Navier Stokes equation. *Physical Review Letters*, 56(14):1505–1508, April 1986.

[7] D. Greenspan. Deterministic computer physics. *International Journal of Theoretical Physics*, 21(6/7):505–523, 1982.

[8] W. D. Hillis. The connection machine: A computer architecture based on cellular automata. *Physica D*, 10:213–228, 1984.

[9] N. Margolus. CAM-8: a computer architecture based on cellular automata. In A. Lawniczak and R. Kapral, editors, *Pattern Formation and Lattice-Gas Automata*. 1993.

[10] Dayanand S. Rajan and Anil M. Shende. A characterization of root lattices. *Discrete Mathematics*, (161):309–314, 1996.

[11] Anil M. Shende. *Digital Analog Simulation of Uniform Motion in Representations of Physical N-Space by Lattice-Work MIMD Computer Architectures*. PhD thesis, SUNY, Buffalo, 1991.

[12] Jerey Yepez. Lattice-gas dynamics, volume I, viscous fluids. Technical Report 1200, Phillips Laboratories, November 1995. Environmental Research Papers.